On homothetic cosmological dynamics

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Abstract

We consider the homogeneous and isotropic cosmological fluid dynamics which is compatible with a homothetic, timelike motion, equivalent to an equation of state $\rho + 3P = 0$. By splitting the total pressure P into the sum of an equilibrium part p and a non-equilibrium part Π, we find that on thermodynamical grounds this split is necessarily given by $p = \rho$ and $\Pi = -\frac{4}{3}\rho$, corresponding to a dissipative stiff (Zel'dovich) fluid.

1   Introduction

Homothetic motions and self-similar spacetimes have been broadly discussed in the literature, mainly from a mathematical but also from a physical point of view (for a recent review see [1]). Applications in astrophysics and cosmology in many cases rely on perfect fluid matter models, but dissipative fluids have been investigated as well [2]. Here, we are interested in a specific aspect of the cosmological dynamics, namely in the imperfect fluid equations of state which are compatible with the existence of a timelike homothetic vector.

Timelike symmetries are not generally expected to exist in the expanding universe. A timelike Killing vector characterizes a stationary spacetime. But it is known [3, 4] that under certain circumstances a *conformal*, timelike symmetry is possible in a Friedmann-Lemaître-Robertson-Walker (FLRW) universe. The

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existence of a conformal timelike Killing vector is closely related to "global" equilibrium properties of perfect fluids. For massless particles the condition for global equilibrium requires the quantity u^a/T, where u^a is the fluid 4-velocity and T is the fluid temperature, to be a conformal Killing vector (CKV). This CKV is compatible with the cosmological expansion for an ultra-relativistic equation of state. Any deviation from the latter will destroy the conformal symmetry. In other words, the conformal symmetry singles out a perfect fluid with the equation of state for radiation. Apparently, this also implies that a conformal symmetry is incompatible with a non-vanishing entropy production. Only the conformal symmetry of an "optical" metric, in which an effective refraction index of the cosmic substratum characterizes specific internal interactions that macroscopically correspond to a negative pressure contribution, may be compatible with the production of entropy [5, 6, 7]. In the present paper we want to point out a different feature of the connection between symmetries and entropy production which, however, also relies on a dissipative scalar pressure.

The cosmological principle restricts the energy momentum tensor T^{ik} of the cosmic substratum to be of the structure

$$T^{ik} = \rho u^i u^k + P h^{ik} \ . \tag{1}$$

Here, ρ is the energy density measured by an observer comoving with the fluid four-velocity u^i which is normalized by $u^a u_a = -1$. The quantity $h_{ik} \equiv g_{ik} + u_i u_k$ is the spatial projection tensor with $h_{ik} u^k = 0$. The total pressure P is the sum

$$P = p + \Pi \tag{2}$$

of an equilibrium part $p > 0$ and a non-equilibrium part $\Pi \leq 0$ which is connected with entropy production. A perfect fluid is characterized by $\Pi = 0$. A scalar, viscous pressure is the only entropy producing phenomenon which is compatible with spatial homogeneity and isotropy. The idea of our paper is to impose a homothetic symmetry requirement on the dynamics of this medium and to ask for the admissible equations of state for p and Π. To this purpose we shall combine the homothetic, timelike dynamics in a Friedmann-Lemaître-Robertson-Walker (FLRW) universe with thermodynamic considerations within the latter. It will turn out that for $P = \alpha\rho$ and $p = w\rho$, where α and w are constants, we have necessarily $\alpha = -1/3$ (cf. [8, 1]) and the split $P = p + \Pi$ is uniquely given by $w = 1$, corresponding to a dissipative fluid with $p = \rho$ and $\Pi = -(4/3)\rho$. An equation of state $P = -1/3\rho$ has been associated with cosmic string matter [9], while $p = \rho$ characterizes a stiff (Zel'dovich) fluid. This means, under the condition of homothetic symmetry a substance with $P = -1/3\rho$ is dynamically equivalent to dissipative stiff matter. This feature of having two interpretations for the same overall equation of state is a property of the homothetic symmetry only and does not hold for a general conformal symmetry which singles out an equation of state $P = p = \rho/3$, i.e., $\Pi = 0$.

In section 2 we briefly recall relevant thermodynamic relations. Section 3 is devoted to the conformal symmetry in the expanding universe while the

homothetic motion is characterized in section 4. Section 5 summarizes our results.

2 Thermodynamic relations

In this section we recall basic thermodynamic relations which will be relevant for the symmetry considerations below. Energy-momentum conservation $T^{ik}_{;k} = 0$ provides us with

$$\dot{\rho} + \Theta \left(\rho + P \right) = 0 \ , \tag{3}$$

where the quantity $\Theta \equiv u^i_{;i}$ is the fluid expansion and $\dot{\rho} \equiv \rho_{,i} u^i$. The particle number balance is

$$N^i_{;i} = \dot{n} + \Theta n = n\Gamma \ , \tag{4}$$

where $N^i = nu^i$ is the particle number flow vector and n is the particle number density. For $\Gamma > 0$ we have particle creation. $\Gamma = 0$ corresponds to particle number conservation. The possible origin of a non-vanishing particle production rate Γ, e.g., a phenomenological description of quantum particle production out of the gravitational field [10, 11, 12, 13, 14, 15] is not relevant here. In the present context the production rate Γ is *not* given by by microphysical considerations but it follows from the (homothetic) symmetry condition (see below). The imposed symmetry may require a certain production rate to be realized at all. Alternatively, as we shall see, the requirement $\Gamma = 0$ under the condition of a homothetic motion leads to an increase in the entropy per particle. Dynamically, both effects are equivalent. From the Gibbs equation

$$T\mathrm{d}s = \mathrm{d}\frac{\rho}{n} + p\mathrm{d}\frac{1}{n} \ , \tag{5}$$

where s is the entropy per particle, it follows that

$$nT\dot{s} = \dot{\rho} - \left(\rho + p \right) \frac{\dot{n}}{n} \ . \tag{6}$$

Using here the balances (3) with (2) and (4) yields

$$nT\dot{s} = -\Theta\Pi - \left(\rho + p \right)\Gamma \ . \tag{7}$$

From the Gibbs-Duhem relation

$$\mathrm{d}p = \left(\rho + p \right)\frac{\mathrm{d}T}{T} + nT\mathrm{d}\left(\frac{\mu}{T} \right) \ , \tag{8}$$

where μ is the chemical potential, we obtain

$$\left(\frac{\mu}{T} \right)^{\cdot} = \frac{\dot{p}}{nT} - \frac{\rho + p}{nT}\frac{\dot{T}}{T} \ . \tag{9}$$

The above set of equations has to be supplemented by (explicitly not yet known) equations of state which are assumed to have the general form

$$p = p\left(n, T \right) \ , \qquad \rho = \rho\left(n, T \right) \ , \tag{10}$$

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i.e., particle number density and temperature are taken as the independent thermodynamical variables. Differentiating the latter relation and using the balances (3) and (4) and relation (7) provides us with an evolution law for the temperature:

$$\frac{\dot{T}}{T} = -(\Theta - \Gamma)\frac{\partial p}{\partial \rho} + \frac{n\dot{s}}{\partial \rho / \partial T} \; , \tag{11}$$

where the abbreviations

$$\frac{\partial p}{\partial \rho} \equiv \frac{(\partial p / \partial T)_n}{(\partial \rho / \partial T)_n} \; , \qquad \frac{\partial \rho}{\partial T} \equiv \left(\frac{\partial \rho}{\partial T}\right)_n \; ,$$

have been used, as well as the general relation

$$\frac{\partial \rho}{\partial n} = \frac{\rho + p}{n} - \frac{T}{n}\frac{\partial p}{\partial T} \; , \tag{12}$$

which follows from the fact that the entropy is a state function. Together with the particle number balance (4) the entropy flow vector $S^a = nsu^a$ gives rise to the following expression for the entropy production density:

$$S^a = nsu^a \quad \Rightarrow \quad S^a_{;a} = ns\Gamma + n\dot{s} \; . \tag{13}$$

Entropy may be produced either by an increase in the number of particles or by an increase in the entropy per particle. Both cases will play a role in our analysis.

3   Conformal symmetry

The condition for the quantity $\xi^i \equiv u^i / T$ to be a CKV is

$$\pounds_\xi g_{ik} = \phi g_{ik} \; . \tag{14}$$

This condition implies

$$\phi = \frac{2}{3}\frac{\Theta}{T} \tag{15}$$

and

$$\frac{\dot{T}}{T} = -\frac{\Theta}{3} \; . \tag{16}$$

In the homogeneous and isotropic case under consideration here, the Raychaudhuri equation for the expansion scalar reduces to

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \frac{\kappa}{2}(\rho + 3P) = 0 \; , \tag{17}$$

with $\kappa = 8\pi G$, where G is Newton's gravitational constant. The derivative of ϕ in (15) then becomes

$$\dot{\phi} = -\frac{\kappa}{3}\frac{\rho + 3P}{T} \; . \tag{18}$$

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Combining the general temperature law (11) with relation (16) yields

$$\Gamma + \frac{n\dot{s}}{\partial p / \partial T} = \left(1 - \frac{1}{3}\frac{\partial \rho}{\partial p}\right)\Theta \; , \tag{19}$$

with Θ being related to the conformal factor by Eq. (15).

Assuming now an overall equation of state $P = \alpha\rho$, the Lie-derivatives of the relevant quantities become

$$\pounds_\xi \rho = -\frac{3}{2}\left(1 + \alpha\right)\phi\rho \; , \qquad \pounds_\xi P = -\frac{3}{2}\left(1 + \alpha\right)\phi P \; , \tag{20}$$

$$\pounds_\xi u_i = \frac{1}{2}\phi u_i \; , \qquad \pounds_\xi u^i = -\frac{1}{2}\phi u^i \; , \tag{21}$$

and

$$\pounds_\xi T_{ab} = -\frac{1}{2}\left(1 + 3\alpha\right)\phi T_{ab} \; . \tag{22}$$

Taking into account that (for $\mu = 0$)

$$s = \frac{\rho + p}{nT} \; , \tag{23}$$

and assuming additionally $p = w\rho$ with a constant w, we have also

$$\pounds_\xi s = -\frac{3}{2}\frac{\alpha - w}{1 + w}\phi s \; . \tag{24}$$

For the standard case $\Gamma = \dot{s} = 0$ the relation (19) is satisfied for $p = \rho/3$. This corresponds to the well-known fact that for a perfect fluid the conformal symmetry condition (14) singles out ultra-relativistic matter. However, one may also ask the inverse question, namely: Does the assumption of a total equation of state $P = \rho/3$ necessarily imply $\Pi = 0$? We shall investigate this for the cases $\dot{s} = 0$ and $\Gamma = 0$ separately. For the case $\dot{s} = 0$ we obtain from Eq. (7) that (cf. [15, 16, 17])

$$\Gamma = -\frac{\Pi}{\rho + p}\Theta \tag{25}$$

is valid which, together with (19), leads to

$$-\Pi = \left(1 - \frac{1}{3}\frac{\partial \rho}{\partial p}\right)(\rho + p) \; , \tag{26}$$

or

$$\rho + P = \frac{1}{3}\frac{\partial \rho}{\partial p}(\rho + p) \; . \tag{27}$$

Introducing here $P = \rho/3$ and $p = w\rho$, relation (27) can only be satisfied for $w = 1/3$, which indeed is equivalent to $\Pi = 0$. For the second special case $\Gamma = 0$ we have from (7)

$$n\dot{s} = -\frac{\Theta\Pi}{T} \; , \tag{28}$$

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which combined with (19) leads to

$$- \Pi = T \frac{\partial}{\partial T} \left[p - \frac{1}{3} \rho \right] \ . \tag{29}$$

With $P = \rho/3$ and $p = w\rho$, as well as with Eq. (12) we find that $\rho \propto nT$. Applying additionally the Gibbs-Duhem equation (8) and the second equation of state in (10), we end up again with $w = \alpha = 1/3$ and $\Pi = 0$. Consequently, in none of these cases the conformal symmetry is compatible with a dissipative fluid configuration.

4   Homothetic motion

Now we assume the conformal factor to be constant, i.e., a homothetic motion. According to (18) this implies

$$\phi = \text{const} \quad \Rightarrow \quad \rho + 3P = 0 \ . \tag{30}$$

An equation of state $P = -\rho/3$ characterizes cosmic string matter [9]. The Raychaudhuri equation (17) in such a case reduces to

$$\dot{\Theta} + \frac{1}{3} \Theta^2 = 0 \ . \tag{31}$$

Introducing here the scale factor a of the Robertson-Walker metric by

$$\Theta \equiv 3H \equiv 3 \frac{\dot{a}}{a} \ , \tag{32}$$

where H is the Hubble-parameter, results in

$$\Theta \propto \frac{1}{a} \quad \Rightarrow \quad a \propto t \ . \tag{33}$$

Under the homothetic condition the energy balance (3) yields

$$\rho \propto \frac{1}{a^2} \ . \tag{34}$$

Since from (16) with (32) one has $T \propto a^{-1}$, this implies $\rho \propto T^2$. The relations (16), (32) and (33) are consistent with $\phi \propto \Theta/T = \text{const}$. Furthermore, (33) and (34) are consistent with Friedmann's equation

$$8\pi G \rho = \frac{\Theta^2}{3} + 3 \frac{k}{a^2} \ , \tag{35}$$

where $k = 0, \pm 1$. Again we consider the cases $\dot{s} = 0$ and $\Gamma = 0$ separately. From Eq.(27) (valid for $\dot{s} = 0$) we obtain

$$\frac{\partial \rho}{\partial p} \left(1 + \frac{p}{\rho} \right) = 2 \ . \tag{36}$$

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In this case $p = w\rho$ leads to $w = 1$, i.e., the equilibrium equation of state is that for stiff matter. Consequently, we have

$$\Pi = -\frac{4}{3}\rho \ . \tag{37}$$

This means, different from the conformal case, *the homothetic symmetry is compatible with a dissipative fluid configuration.* The set (20) - (22) of Lie-derivatives becomes

$$\pounds_\xi \rho = -\phi\rho \ , \qquad \pounds_\xi P = -\phi P \ , \qquad \pounds_\xi T_{ab} = 0 \ , \qquad \pounds_\xi s = \phi s \ . \tag{38}$$

From (29) with $-\Pi = p + \frac{\rho}{3}$ and $\rho \propto T^2$ we find $p = \rho$ also in the case $\Gamma = 0$, $\dot{s} \neq 0$. This means, Eq. (37) is valid for both the cases considered here. For $\dot{s} = 0$ one has

$$\Gamma = 2H = \phi T \ , \tag{39}$$

i.e., the production rate is required to be twice the Hubble rate H. For the case $\Gamma = 0$ it follows

$$\frac{\dot{s}}{s} = 2H \tag{40}$$

for the fractional change of the entropy per particle. To complete the thermodynamical description of our system we realize that $\Pi = -4/3\rho$ in (29) amounts to

$$\frac{4}{3}\rho = \frac{2}{3}T\frac{\partial\rho}{\partial T} \quad \Rightarrow \quad T\frac{\partial\rho}{\partial T} = 2\rho \ , \tag{41}$$

which is consistent with $\rho \propto T^2$. Using the last equation of (41) in the expression (12) for $\partial\rho/\partial n$ we find

$$\frac{\partial\rho}{\partial n} = 0 \ . \tag{42}$$

All this is consistent with the Gibbs-Duhem equation (8) for $\mu = 0$ since

$$\mathrm{d}p = (\rho + p)\frac{\mathrm{d}T}{T} \quad \Rightarrow \quad \mathrm{d}\rho = 2\rho\frac{\mathrm{d}T}{T} \ . \tag{43}$$

If we formally define a coefficient of bulk viscosity according to $\Pi = -\zeta\Theta$, we obtain

$$\zeta = \frac{8}{9}\frac{\rho}{\phi T} = \frac{m_P^2}{6\pi}H\left[1 + \frac{k}{a^2 H^2}\right] \ . \tag{44}$$

Here we have applied equations (37), (15) and (35) and introduced the square of the Planck mass $m_P^2 = G^{-1}$. Because of the dependence (33) one has $a^2 H^2 = $ const in the last expression of (44).

5 Conclusion

Given that the total pressure P of the homogeneous and isotropic cosmic medium splits into an equilibrium part p and a non-equilibrium contribution Π, and at

the same time u^i/T is a homothetic vector, the equations of state are necessarily $p = \rho$ and $\Pi = -(4/3)\rho$. This corresponds to a dissipative Zel'dovich fluid with an effective bulk viscosity coefficient $\zeta \propto \rho^{1/2}$.

Acknowledgments

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